A NOVEL ENERGY RELATION IN EIGEN MODES OF TRANSMISSION LINE AND ITS APPLICATION TO THE DERIVATION OF VARIATIONAL EXPRESSION FOR PROPAGATION CONSTANT

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ABSTRACT

A novel energy relation in eigen modes of transmission line is herein proposed and the variational expressions for propagation constant of various transmission lines, which have never been treated in variational form, can be systematically derived through this relation.

Introduction

Propagation constants of eigen modes traveling along a uniform transmission line are important parameters in analyzing the characteristics of transmission line. The precise explicit expression for propagation constant, however, cannot be obtained, except for very simple structure. Therefore, the variational expression for propagation constant is very useful when the structure of transmission line is complex.

Although some variational expressions have already been obtained[1-5], we will provide, here, a new derivation of the variational expression for propagation constant from a new energy relation in eigen modes of transmission line.

A Novel Energy Relation

It is known [4] that the time-average electric energy is equal to the time-average magnetic energy for the eigen mode propagating along the loss-free transmission line, and the variational expression for propagation constant can be derived from this principle. The same conclusion has been obtained for the loss-free cavity resonator and its resonance frequencies[3].

The above conclusion, however, is not valid for the evanescent eigen modes of loss-free transmission line. The magnetic energy is greater than the electric energy for the evanescent TE modes in the homogeneous waveguide, whereas the electric energy is greater than the magnetic energy for the evanescent TM modes.

It should be noted, however, that while the energy in the above discussion is regarded either as the electric energy or the magnetic energy, the axial field components energy can be proved (See Appendix-1) to be equal to the transverse field components energy for the evanescent eigen modes of loss-free transmission line. That is,

$$\iint H_zB_z^*ds = \iint H_tB_t^*ds + \iint E_tD_t^*ds \quad \text{(TE modes)} \quad (1)$$

$$\iint E_zD_z^*ds = \iint H_tB_t^*ds + \iint E_tD_t^*ds \quad \text{(TM modes)} \quad (2)$$

in which the astersks stand for the complex conjugate, z-axis for the axis of transmission line, and the subscript t for the transverse components. The surface integral is performed over the cross sectional area of transmission line.

Furthermore, the following integral may be expected to vanish for any eigen mode:

$$I=\iint [H_tB_t-H_zB_z]ds - \iint [E_tD_t-E_zD_z]ds = 0 \quad (3)$$

Actually, the integral I can be proved to become zero by the application of Maxwell's equations. (See Appendix-2) Equation (3) demonstrates the new energy relation in eigen modes of transmission line. The complex conjugate sign is dropped in Eq.(3), because no useful formula containing it can exist for the lossy transmission line. Furthermore it is assumed in the above discussion that the perfect electric or magnetic wall encloses the transmission line. This means that the power-flow toward the transverse direction vanishes along the boundary C enclosing the transmission line. If the transverse power-flow through C occurs, Eq.(3) can be modified as follows:

$$I=\iint [H_tB_t-H_zB_z-E_tD_t+E_zD_z]ds + \frac{j}{\omega}\int n\cdot[E_t\times H_z-E_z\times H_t]dl = 0 \quad (4)$$

where the line integral in Eq.(4) corresponds to the power-flow through the boundary C and n is the outward normal unit vector.

Even when the materials composing the guide are dissipative, inhomogeneous, anisotropic and non-reciprocal, the integral I given by Eq.(4) always vanishes for any modes. Moreover, the equation also allows for the discontinuous change of materials in the transverse plane as well as for the imperfect boundary wall. (See Appendix-2)

The Variational Expressions for Propagation Constant

We shall show that the variational expressions for propagation constant can be systematically derived from Eq.(3) or Eq.(4).

Ferrite-filled Waveguide†

We will investigate the waveguide involving ferrite materials magnetized perpendicularly to the broad side plane, as the first example. For simplicity we will study only the modes of which single electric field component is Ey.

Let Ey be of the form:

$$E_y=f(x)\exp(-\gamma z) \quad (5)$$

The other field components are written in terms of f(x) by the use of Maxwell's equations.

$$B_x=-\frac{\gamma}{j\omega}f(x)\exp(-\gamma z) \quad (6)$$

$$B_z=-\frac{f'(x)}{j\omega}\exp(-\gamma z) \quad (7)$$

$$H_x=\frac{j\gamma\mu f(x)-\kappa f'(x)}{\omega\mu e}\exp(-\gamma z) \quad (8)$$

$$H_z=\frac{\gamma\kappa f(x)+j\mu f'(x)}{\omega\mu e}\exp(-\gamma z) \quad (9)$$

where μ and κ mean the diagonal and off-diagonal

† Collin[4] has been treated this problem in a similar way but his treatment of evanescent modes is not clear.

elements of ferrite tensor permeability, respectively and $\mu e = (\mu^2-\kappa^2)/\mu$ and ω is the operating angular frequency and $f'(x)$ is the first derivative of $f(x)$.

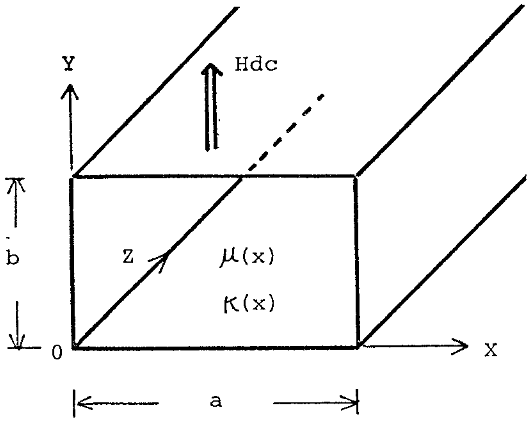

Fig.1 Ferrite-filled Waveguide.

Therefore, the integral I given by Eq.(3) can be written in terms of f(x) as follows:

$$I=b\int_0^a(\omega^2\mu e)^{-1}[-(\omega^2\mu e\varepsilon+\gamma^2)f(x)-2j\gamma\frac{\kappa}{\mu}f(x)f'(x)+f'(x)^2]dx \quad (10)$$

The function f(x) making I stationary, must satisfy the following differential equation:

$$\frac{d}{dx}[\frac{f'(x)}{\mu e}-j\gamma\frac{\kappa f(x)}{\mu\ \mu e}]+j\gamma\frac{\kappa}{\mu\mu e}f'(x)+\frac{\gamma^2+k^2}{\mu e}f(x)=0 \quad (11)$$

Eq.(11) is also directly deduced from Maxwell's equations. Futhermore, the correct function f(x) satisfying Eq.(11), makes the integral I vanish.

Inversely, the variational expression for propagation constant γ can be obtained, imposing the restriction that Eq.(10)=0.

$$\gamma=\frac{j\int_0^a\frac{a\kappa ff'}{\mu\mu e\omega^2}dx\pm\sqrt{-(\int_0^a\frac{a\kappa\ ff'}{\mu\omega^2\mu e}dx)-\int_0^a\frac{f^2}{\omega^2\mu e}dx\int_0^a\frac{af'^2-\omega^2\mu e\varepsilon f^2}{\omega^2\mu e}dx}}{\int_0^a(\omega^2\mu e)^{-1}f^2dx} \quad (12)$$

Waveguide with Wall Impedance

The next example is the waveguide with wall impedance, initially investigated by Kurokawa[5]. For this problem, Eq.(4) must be employed in deriving the variational expression, because the transverse power-flow occurs at the boundary of the waveguide.

We will seek, herein, the functional of which trial function is the transverse electric field Et(x,y). For this reason, the other field components must be written in terms of Et(x,y) by the application of Maxwell's equations.

$$Hz=-\frac{\nabla\times Et}{j\omega\mu} \quad (13)$$

$$Ez=\frac{\nabla\cdot\varepsilon Et}{\gamma\varepsilon} \quad (14)$$

$$Ht=\frac{k\times(\nabla\times\frac{1}{\mu}\nabla\times Et-\omega^2\varepsilon Et)}{j\omega\gamma} \quad (15)$$

For brevity, the materials involved in the waveguide are assumed to be isotropic. The wall impedance Z_1 and Z_2 are defined as follows:

$$Z_1Ht=n\times Ez \quad (\text{ on C }) \quad (16)$$

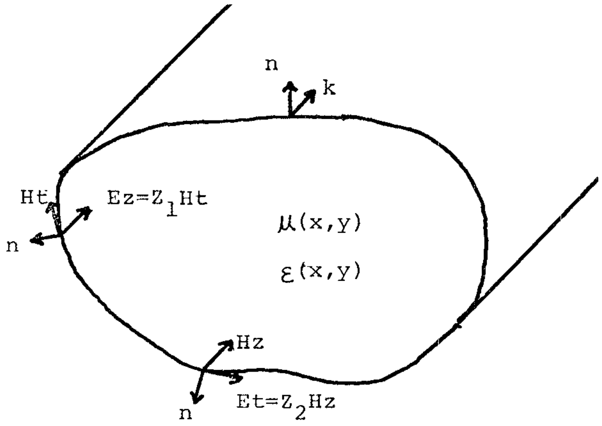

Fig.2 Waveguide with wall impedance.

$$Z_1Hz=n\times Et \quad (\text{ on C }) \quad (17)$$

Substituting Eqs.(13)-(17) into Eq.(4), we have

$$I=\iint[-\frac{\mu}{\gamma^2\omega^2}(\nabla\times\frac{1}{\mu}\nabla\times Et-\omega^2\varepsilon Et)^2+\frac{1}{\omega^2\mu}(\nabla\times Et)^2-\varepsilon Et^2+\frac{1}{\gamma^2\varepsilon}(\nabla\cdot\varepsilon Et)^2]ds-\int[\frac{jZ_2}{\omega^3}(\frac{1}{\mu}\nabla\times Et)^2-\frac{jZ_1}{\omega^3\gamma^2}(\nabla\times\frac{1}{\mu}\nabla\times Et-\omega^2\varepsilon Et)^2]dl \quad (18)$$

Upon imposing I=0, the propagation constant γ can be solved from Eq.(18).

$$\gamma^2=\{\iint[\mu(\nabla\times\frac{1}{\mu}\nabla\times Et-\omega^2\varepsilon Et)^2-\frac{\omega^2}{\varepsilon}(\nabla\cdot\varepsilon Et)^2]ds+\int\frac{Z_1}{j\omega}(\nabla\times\frac{1}{\mu}\nabla\times Et-\omega^2\varepsilon Et)^2dl\}/\{\iint[\frac{1}{\mu}(\nabla\times Et)^2-\omega^2\varepsilon Et^2]ds+\int\frac{Z_2}{j\omega}(\frac{1}{\mu}\nabla\times Et)^2dl\} \quad (19)$$

Eq.(19) is equal to the variational expression initially given by Kurokawa[5]. However, he gave no way how to derive this variational expression.

Conclusion

The novel energy and power-flow relation in eigen modes of uniform transmission line proposed here, has a wide applicability. The variational expression for propagation constant can be systematically derived from this relation. Futher application is now being done.

References

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Appendix

1. Proof of Eqs.(1) and (2).

For TE modes, the single axial field component is Hz, and the other field components are written in terms of Hz, by the application of Maxwell's equations.

$$Et=\frac{j\omega\mu}{k^2}\, k\times\nabla Hz \quad (A1)$$

$$Ht=-\frac{\gamma}{k^2}\,\nabla Hz \quad (A2)$$

where

$$k^2=\gamma^2+\omega^2\mu\varepsilon \quad (A3)$$

Hence, we have

$$Wet=\iint\varepsilon|Et|^2$$
$$=\mu\frac{\omega^2\mu\varepsilon}{k^4}\iint(\nabla Hz)^2ds \quad (A4)$$

$$Wmt=\iint\mu|Hz|^2ds$$
$$=\mu\frac{|\gamma|^2}{k^4}\iint(\nabla Hz)^2ds \quad (A5)$$

$$Wmz=\iint\mu|Hz|^2ds \quad (A6)$$

Furthermore, Hz must obey the Helmholtz equation:

$$\nabla^2Hz + k^2Hz=0 \quad (A7)$$

and satisfy the boundary condition:

$$\frac{\partial Hz}{\partial n} = 0 \quad (A8)$$

Substituting Eqs.(A7) and (A8) into Eq.(A6), the following equation can be obtained.

$$Wmz= -\frac{\mu}{k^2}\iint Hz\nabla^2Hzds$$
$$= -\frac{\mu}{k^2}\left[\int Hz\frac{\partial Hz}{\partial n}dl - \iint(\nabla Hz)^2ds\right]$$
$$=\frac{\mu}{k^2}\iint(\nabla Hz)^2ds \quad (A9)$$

Therefore, when $k^2 - \omega^2\mu\varepsilon<0$,

$$Wmz + Wmt = Wet \quad (A10)$$

while $k^2 - \omega^2\mu\varepsilon>0$,

$$Wmz = Wmt + Wet \quad (A11)$$

On the other hand, Ez is regarded as a generating function for TM modes, and the same conclusion can be easily deduced.

2. Proof of Eqs.(3) and (4).

Maxwell's equations are resolved into two components, the transverse component and longitudinal component.

$$\nabla t\times Ez - \gamma\; k\times Et = -j\omega Bt \quad (A12)$$

$$\nabla t\times Et= -j\omega Bz \quad (A13)$$

$$\nabla t\times Hz - \gamma\; k\times Ht = j\omega Dt \quad (A14)$$

$$\nabla t\times Ht=j\omega Dz \quad (A15)$$

where, k is a unit vector in the longitudinal direction. From Eqs.(A12)-(A15), we have

$$HtBt-EtDt= \frac{j}{\omega}[Hz\cdot\nabla t\times Ez + Et\cdot\nabla t\times Hz] \quad (A16)$$

$$HzBz-EzDz= \frac{j}{\omega}[Hz\cdot\nabla t\times Et + Ez\cdot\nabla t\times Ht] \quad (A17)$$

Using the vector identity:

$$\int n\cdot(A\times B)dl=\iint B\cdot\nabla\times Ads-\iint A\cdot\nabla\times Bds \quad (A18),$$

we have also the following equations:

$$\int n\cdot(Ht\times Ez)dl=\iint Ez\cdot\nabla\times Htds-\iint Ht\cdot\nabla\times Ezds$$
$$=\iint Ez\cdot\nabla t\times Htds-\iint Ht\cdot\nabla t\times Ezds \quad (A19)$$

$$\int n\cdot(Et\times Hz)dl=\iint Hz\cdot\nabla\times Etds-\iint Et\cdot\nabla\times Hzds$$
$$=\iint Hz\cdot\nabla t\times Etds-\iint Et\cdot\nabla t\times Hzds \quad (A20)$$

Hence, we have Eq.(4). That is,

$$\iint[HtBt-HzBz]ds-\iint[EtDt-EzDz]ds$$
$$+\frac{j}{\omega}\int n\cdot(Et\times Hz-Ez\times Ht)dl=0 \quad (A21)$$

Especially, if the perfect electric or magnetic wall exist at the boundary of transmission line, the line-integral in Eq.(A21) will vanish.